Assumptions in equilibrium analysis and experimentation in unsaturated soil

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Abstract

The fundamental considerations and assumptions in laboratory equilibrium tests on soils in the triaxial cell are examined using the principles of virtual work and thermodynamics. Compliance with the laws of thermodynamics is essential and the first law of thermodynamics necessitates that the work equation, thus the stresses, may be derived from the thermodynamic potential. At equilibrium the thermodynamic potential is a minimum and it is shown that this can be written in a similar form for both isotropic and anisotropic loading conditions but for the latter the mean stress replaces the isotropic pressure. The significance of the extensive variable terms making up the thermodynamic potential is also described. The analysis is applicable to soils at any degree of saturation.

Isotropic loading conditions

The thermodynamic concepts of internal energy and entropy may without modification be applied to soils. Under equilibrium conditions the internal energy U for an isotropically loaded specimen of volume V may be written as,

[1] $U = TS - pV$

where, T is the absolute temperature of the specimen
S is the entropy of the specimen
p is the applied pressure

This assumes no chemical potential, which could lead to osmotic suction, and ignores the gravitational field. For any subsequent change in the variables of state S and V, maintaining the respective conjugate parameters T and p constant, the internal energy given by Equation [1] represents the Euler thermodynamic potential (Sposito, 1981, Callen, 1965). For an infinitesimal change in thermal and work energy as a result of changes dS and dV,

[2] $dU = dQ + dW = TdS - pdV$

where, dQ is the heat added to the specimen

$dW = -pdV$ is the virtual work done to the specimen

dS is the increase in entropy

dV is the change in volume ($dV<0$ for compression giving $dW>0$)

dU is the change in internal energy

Equation [2] is written to reflect the situation where work is done to the specimen and to be consistent with volumetric compression, and accordingly length and radius reduction, corresponding to positive strain increments for positive compressive stresses.

The work analysis considers a soil specimen in the triaxial cell and treats the specimen as a mass with no distinction as to its degree of saturation or composition and is thus applicable to soils at any degree of saturation. The analysis thus considers only the applied total stresses and does not consider 'effective' interparticle stresses or fluid pressures in the soil specimen and says nothing about the specimen's history.

The triaxial cell is taken as acting as a thermal and volume 'reservoir' with the cell wall taken as a rigid adiabatic barrier allowing no heat transfer with the surroundings. The cell and the specimen contained therein are referred to as the system. Two extremes of soil test may be carried out on the soil specimen. The specimen may be allowed to exchange air and water with an external measuring system under drained conditions, or an undrained test under closed conditions with no matter exchange may be carried out. The latter is considered here and it is assumed that at equilibrium and during the infinitesimal changes considered that there is no transfer of mass or heat from or to the system and in particular there is no exchange with external pressure or volume measurement devices. The cell reservoir (water) applies an isotropic loading to the soil specimen p_w contained in an impermeable sheath, which prevents direct contact and mass interchange between the reservoir and the soil, but imparts no loading. The soil specimen has a responsive pressure p_m. Under the above conditions, the internal energy change for the system dU is given by Equation [3] (Sposito, 1981).

[3] $$dU = dU_m + dU_w = dQ_m + dW_m + dQ_w + dW_w = T_m dS_m - p_m dV_m + T_w dS_w - p_w dV_w$$

where, dU_m and dU_w are the changes in internal energy of the soil (mass) and reservoir (water in cell) respectively

$dQ_m = T_m dS_m$ is the change in heat of the soil

$dQ_w = T_w dS_w$ is the change in heat of the reservoir

$dW_m = p_m dV_m$ is the work done on the soil

$dW_w = p_w dV_w$ is the work done on the reservoir

T_m and T_w are the absolute temperatures of the soil and reservoir respectively

dS_m and dS_w are the changes in entropy of the soil and reservoir respectively

dV_m and dV_w are the changes in volumes of the soil mass and reservoir respectively

It is assumed in the analysis that the infinitesimal changes are reversible and are thus essentially virtual.

There is no change in total entropy of the system as the cell wall acts as an adiabatic barrier, thus $dS_m = -dS_w$. There is also no change in volume of the system for a rigid cell wall thus, $dV_m = -dV_w$. In addition, assuming the establishment of thermal equilibrium within the system $T_m = T_w$. Under these conditions Equation [3] may be written as,

[4] $dU = -(p_m - p_w)\, dV_m$

Equation [4] describes a virtual process with an infinitesimal change in the soil volume dV_m. For equilibrium $dU = 0$ as there is a requirement for the thermodynamic potential U given by Equation [1] to be a minimum at equilibrium. The equation confirms that under isotropic loading conditions, proved the assumptions outlined are satisfied, a prime requirement in comparing equilibrium conditions in the triaxial cell with theoretical predictions is that the pressure imposed by the water in the cell p_w is balanced by the pressure exerted within the soil sample p_m.

Anisotropic loading conditions

Now consider the case of a specimen of height h, cross sectional area A and radius r subject to a more general class of loading with a total axial stress σ_1 and a cell pressure σ_3 such that the mean applied stress $p = (\sigma_1 + 2\sigma_3)/3$ and the deviator stress $q = (\sigma_1 - \sigma_3)$. The work equation is give by $dW = -\sigma_3 dV - (\sigma_1 - \sigma_3)Adh$. The coincidence of the principal axes of stress and strain-increment is assumed. It is also assumed that the application of a deviatoric stress from the ram in the triaxial cell does not influence the thermodynamics of the system. For an infinitesimal, virtual transfer of thermal and work energy under similar conditions to the assumptions for isotropic loading, the internal energy change for the system dU is given by:

[5] $dU = dU_m + dU_w = dQ_m + dW_m + dQ_w + dW_w = T_m dS_m - \sigma_{3m} dV_m - (\sigma_{1m} - \sigma_{3m})\, Adh_m + T_w dS_w - \sigma_{3w} dV_w - (\sigma_{1w} - \sigma_{3w}) Adh_w$

where, σ_{1m} and σ_{1w} are the total axial stresses of the soil and cell loading system respectively

σ_{3m} and σ_{3w} are the total lateral stresses of the soil and cell loading system respectively

dh_m and dh_w are the axial compression of the soil specimen and displacement of the cell loading system respectively

As previously, for no change in total entropy of the system $dS_m = -dS_w$ and for no net change in volume of the system, $dV_m = -dV_w$. It is also necessary to assume compatibility of axial displacement, $dh_m = -dh_w$. In addition, $T_m = T_w$ if thermal equilibrium of the system is established and, as previously, $dU = 0$. Under these conditions, Equation [5] may be written as,

[6] $$0 = -(\sigma_{3m} - \sigma_{3w})dV_m - [(\sigma_{1m} - \sigma_{3m}) - (\sigma_{1w} - \sigma_{3w})]\, Adh_m$$

Dividing throughout by the volume of the specimen V_m, gives

[7] $$0 = (\sigma_{3m} - \sigma_{3w})(\varepsilon_{lm} + 2\varepsilon_{rm}) + [(\sigma_{1m} - \sigma_{3m}) - (\sigma_{1w} - \sigma_{3w})]\, \varepsilon_{lm}$$

where, for stress and strain positive in compression (Schofield and Wroth, 1968),

$\varepsilon_{vm} = -dV_m/V_m = (\varepsilon_{lm} + 2\varepsilon_{rm})$
$\varepsilon_{lm} = -dh_m/h$
$\varepsilon_{rm} = -dr_m/r$
dr_m is the change in radius of the soil specimen

Re-arranging [7] it is readily shown that,

[8] $$0 = (p_m-p_w)(\varepsilon_{lm} + 2\varepsilon_{rm}) + (q_m-q_w)\, 2(\varepsilon_{lm}-\varepsilon_{rm})/3$$

where, $p_m = (\sigma_{1m} + 2\sigma_{3m})/3$
$q_m = (\sigma_{1m} - \sigma_{3m})$
$p_w = (\sigma_{1w} + 2\sigma_{3w})/3$
$q_w = (\sigma_{1w} - \sigma_{3w})$

Equilibrium analysis vertically and radially gives $\sigma_{1m} = \sigma_{1w}$ and $\sigma_{3m} = \sigma_{3w}$ and thus from Equation [8] $p_m = p_w$ and $q_m = q_w$. However, this does not necessarily follow directly from Equation [8] unless the influence of the mean stress and deviator stress in the virtual work equation are treated independently.

Virtual work input and thermodynamic potential

Compliance with the laws of thermodynamics is a required feature of any soil model if it is to be based on sound principles (Houlsby et al, 2005). In accordance with the first law of thermodynamics it is necessary that the work equation thus the stresses can be derived from a thermodynamic potential. In soils this is complicated by the general anisotropic loading conditions. The work input dW_m to the soil specimen per unit soil volume under the anisotropic undrained loading considered is given by,

[9] $$\frac{dW_m}{V_m} = p_m(\varepsilon_{lm} + 2\varepsilon_{rm}) + q_m\frac{2}{3}(\varepsilon_{lm}-\varepsilon_{rm}) = -p_m\frac{dV_m}{V_m} - q_m\frac{2}{3}\left(\frac{dh_m}{h} - \frac{dr_m}{r}\right)$$

In addition, the change in internal energy dU_m may be written as,

[10] $$dU_m = T_m dS_m + dW_m$$

Substituting for dW_m from Equation [9],

$$[11] \quad dU_m = T_m dS_m - p_m\, dV_m - q_m \frac{2}{3} V_m \left(\frac{dh_m}{h} - \frac{dr_m}{r}\right)$$

The associated extensive thermodynamic potential U_m before the increment of virtual work determined from Equation [11] is given by,

$$[12] \quad U_m = T_m S_m - p_m V_m$$

Equation [12] is in the same form as the Euler Equation [1] for isotropic loading but p_m represents the mean stress. There is no term in the potential for the deviator stress as on integration of Equation [11] the deviator strain term reduces to zero. Alternatively, the appropriateness of Equation [12] may be demonstrated by differentiation, but it is not appropriate to merely write $dU_m = T_m dS_m - p_m dV_m$ as this is only true for isotropic loading conditions. Rewriting Equation [12] as [12a],

$$[12a] \quad U_m = T_m S_m - \frac{1}{3}(\sigma_{1m} + 2\sigma_{3m})V_m = T_m S_m - \frac{1}{3}(\sigma_{1m} - \sigma_{3m})Ah_m - \sigma_{3m} V_m.$$

Substituting $A = \pi r_m^2$ and noting that $r_m = Nh_m$ where N is the ratio of radius to height of the specimen, integration correctly leads to $dU_m = T_m dS_m - \sigma_{3m} dV_m - (\sigma_{1m} - \sigma_{3m})Adh_m = T_m dS_m + dW_m$.

The analysis indicates that the mean stress term may be written as a deviator stress term plus an isotropic stress term arising from the cell pressure as shown in Equation [12a]. Thus the thermodynamic potential for anisotropic loading given by Equation [12] leads to the correct work equation. The fact that a term for q_m does not appear in Equation [12] suggests that at equilibrium it is appropriate to treat the mean stress and deviator stress independently, as suggested in relation to Equation [8] in assessing the imposed stresses and stresses in the soil specimen in the triaxial cell.

The terms U_m, S_m and V_m in Equation [12], and thus the thermodynamic potential, are extensive variables (Sposito, 1981). Combining an intensive variable with the conjugate extensive variable such as in the term $p_m V_m$ results in an extensive variable. A property of the extensive variables is that they are 'additive' in the sense that, for example, the total volume of the phases in a soil is the sum of the volumes of the individual phases. Similarly, the term $p_m V_m$ is additive. This is made up of the thermodynamic potential terms $\sigma_{1m} V_m/3$ and $2\sigma_{3m} V_m/3$. The inclusion of a deviator stress term in the thermodynamic potential Equation [12] would appear to violate the principle.

Examination of Equation [12] indicates that enthalpy $H_m = p_m V_m + U_m$ is also an extensive variable and its additive property is used by Murray (2002) to derive an equation describing the stress regime in unsaturated soils under equilibrium conditions.

Equilibrium assumptions and conditions

The foregoing analysis indicates that the conditions in the triaxial cell for compatibility between experimental and theoretical equilibrium are: $p_m = p_w$, $q_m = q_w$, $dV_m = -dV_w$, $T_m = T_w$, $dU_m = -dU_w$ and $dS_m = -dS_w$. This assumes the triaxial cell can be treated as an isolated system where adiabatic conditions exist at the rigid outer cell wall. Equilibrium also necessitates no mass exchange between the soil and the reservoir, no mass loss or gain by the system and no chemical imbalance leading to a chemical potential.

Experimentally, mechanical equilibrium is also described by no further measurable changes within the soil specimen. However, this may not be a necessary requirement for pressure equilibrium and may take longer to become established particularly in unsaturated soils where internal phase pressure and strain interactions may take longer to equilibrate though overall pressure equilibrium may apparently be satisfied. Barden and Sides (1967) concluded that in unsaturated soils there is evidence to suggest that equilibrium in terms of Henry's law may require a considerable time interval, far greater than that in the absence of soil particles.

Equating the applied stresses to those experienced by the soil, q_w applied to the soil specimen is balanced by an equal and opposite q_m from the soil specimen leading to the conclusion that at equilibrium $q_w = (\sigma_{1m} - \sigma_{3m})$. The deviator stress is the net resistance to shearing as a result of interaction of the soil particles. Similarly, there is a balance between the mean stress p_w applied to the soil specimen and an equal and opposite p_m from the soil leading to the conclusion that $p_w = (\sigma_{1m} + 2\sigma_{3m})/3$. This is made up of the net effect of the mean stresses arising from interaction of the soil particles along with other spherical pressures such as the water and air pressure and the spherical phase interaction effects of surface tension, adsorbed water, dissolved air and water vapour. The foregoing describes the normal assumption that the external pressure and loading measurements are directly related to the pressures and stresses in the soil.

Equilibrium is governed by a thermodynamic potential. This is a minimum at equilibrium and can be written in a similar form for both isotropic and anisotropic loading conditions but for the latter the mean stress replaces the isotropic pressure. The significance of the extensive variable terms making up the thermodynamic potential is briefly described and their significance is expanded upon in a separate paper by Murray and Sivakumar in this conference.

References

Barden, L., and Sides, G.R. 1967. The diffusion of air through the pore water of soils. *Proceedings of the 3rd Asian Regional Conference on Soil Mechanics and Foundation Engineering, Israel, Vol. 1*, 135-138.

Callen, H.B. (1965). *Thermodynamics*. John Wiley and Sons, Inc.

Schofield, A.N. and Wroth, C.P. (1968). *Critical state soil mechanics*. London: McGraw-Hill.

Sposito, G (1981). *The thermodynamics of soil solutions*. Oxford Clarendon Press.

Tarantino, A. Mongiovi, L and Bosco, G. 2000. *An experimental investigation on the independent isotropic stress variables for unsaturated soils*. Geotechnique, ***50(****3)*: 275-282.

Houlsby, G.T., Amorosi, A. and Rojas, E. (2005). Elastic moduli of soils dependent on pressure: a hyperelastic formulation, *Gotechnique* ***LV***, No. 5383-392.

Murray, E.J. (2002). An equation of state for unsaturated soils. *Can. Geotech. J.* ***39***, 125-140.